10.5: Ratio & Root Tests

Theorem (Ratio Test):

Assume that the limit $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists

(1) If $\rho < 1$ then $\sum a_n$ converges absolutely

(2) If $\rho > 1$ then $\sum a_n$ diverges

(3) If $\rho = 1$ the test is inconclusive

($\sum a_n$ may conv. or diverge)

Exercise: Read the proof (or better yet, prove it!)

Here's the proof of (1) If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < \rho < 1$

There is an M and an r so that for all $n \geq M$

$|a_{n+1}|/|a_n| < r < 1$

So $|a_{M+1}| < r|a_M|$

$|a_{M+2}| < r|a_{M+1}| < r^2|a_M|$

$\vdots$

$|a_{M+n}| < r^n|a_M|$

$$\Rightarrow \sum_{n=M}^{\infty}|a_n| = \sum_{n=0}^{\infty}|a_{n+M}| < \sum_{n=0}^{\infty} r^n|a_M| = |a_M| \underbrace{\sum_{n=0}^{\infty} r^n}$$

Geometric series with $r<1$

So $\sum_{n=M}^{\infty}|a_n|$ converges by the comparison test.

~~~x~~~

Example:

Does $\sum_{n=1}^{\infty}\frac{7^n}{n!}$ converge?

↳ Tip: seeing a factorial should tell you to try the ratio test.

Sol'n: $a_n = \frac{7^n}{n!}$,

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{7^{n+1}/(n+1)!}{7^n/n!} = \lim_{n\to\infty}\frac{7}{n+1} = 0$$

So $\sum_{n=1}^{\infty}\frac{7^n}{n!}$ converge by the ratio test.
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Example: Does $\sum_{n=0}^{\infty} (-1)^n \frac{n!}{3^n}$ converge?

Sol'n: $a_n = (-1)^n \frac{n!}{3^n}$

So $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!\, 3^n}{(n)!\, 3^{n+1}} = \lim_{n\to\infty} \frac{n+1}{3}$

$= \infty$

$\Rightarrow$ the series diverges.

Example: What does the ratio test say about $\sum_{n=1}^{\infty} n^{-2}$? How about $\sum_{n=1}^{\infty} n^2$?

Sol'n: The ratio test is inconclusive in both cases (why?). But $\sum_{n=1}^{\infty} n^{-2}$ converges (why?) and $\sum_{n=1}^{\infty} n^2$ diverges (why?)

Theorem (Root test)

Assume that the limit $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

exists

(1) If $L<1$, then $\sum a_n$ converges absolutely

(2) If $L>1$, then $\sum a_n$ diverges

(3) If $L=1$, the test is inconclusive.

Exercise: Prove the theorem (very similar to the proof of the ratio test)

Example:

Does $\sum_{n=1}^{\infty} \left(\frac{2n+1}{5n+7}\right)^n$ converge? ← tip: when you see nth power try root test.

Sol'n $\lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+1}{5n+7}\right)^n} = \lim_{n\to\infty} \frac{2n+1}{5n+7} = \frac{2}{5} < 1$

$\Rightarrow$ conv. by root test

Example: Does $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n}$ conv. or div according to the root test?

Soln: $a_n = \left(1+\frac{1}{n}\right)^{-n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-1} = \lim_{n\to\infty} \frac{n}{n+1}$$

$$= 1 \implies \text{Inconclusive!}$$

Example: Does $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$ converge?

Soln: $a_n = \left(1+\frac{1}{n}\right)^{-n^2}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n^2/n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$$

$$= \frac{1}{e} < 1$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2} \text{ converges.}$$

10.6 Power Series

A Power Series with center c is an infinite series

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

$$= a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

Example: $F(x) = 1 + 2(x-7) + 3(x-7)^2 + \cdots$

$$= \sum_{n=0}^{\infty} (n+1)(x-7)^n$$

Since a power series is a series that depends on x, it may converge for certain values of x and diverge for others

Example: $\sum_{n=0}^{\infty} x^n$ converges when $|x| < 1$

and diverges when $|x| \geq 1$.

(why?)

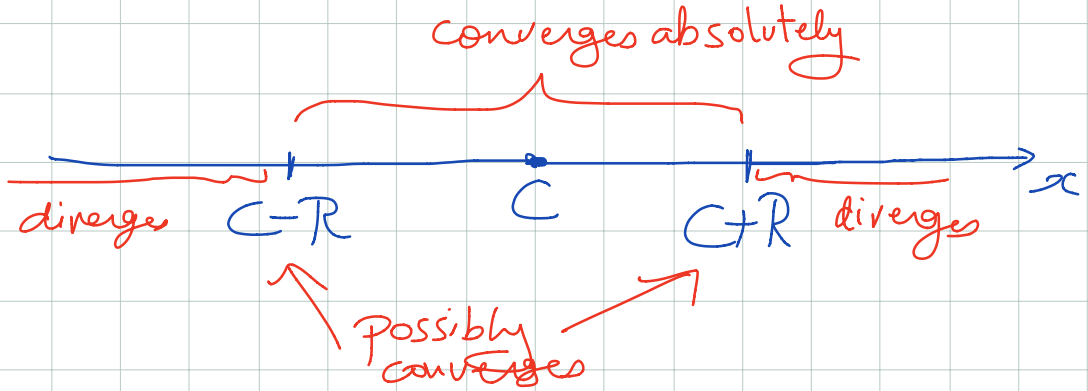
Theorem (Radius of Convergence)

Every Power Series

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

has a radius of convergence R, where either $R \geq 0$ or $R = \infty$.

- If R is finite $F(x)$ converges absolutely when $|x-c| < R$ & diverges when $|x-c| > R$
- If $R = \infty$, then $F(x)$ converges absolutely for all x.



Example: $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} (x-0)^n$ ($a_n = 1$, $c = 0$)

converges when $|x-0| < 1$

and diverges otherwise

Exercise: Read the proof in the book.

Finding the interval of convergence of $F(x)$

Step 1: Find the Radius of conv. R (usually with ratio test)

Step 2: Check the endpoints (when $R \neq 0$ & $R \neq \infty$)

Example: Where does $F(x) = \sum_{n=0}^{\infty} \frac{x^n}{2^n}$ converge

Soln: (1) Let $b_n = \frac{x^n}{2^n}$ and compute

$$\rho = \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}/2^{n+1}}{x^n/2^n}\right| = \lim_{n\to\infty} \left|\frac{x}{2}\right| = \frac{|x|}{2}$$

So $\rho < 1$ whenever $|x|/2 < 1 \iff |x| < 2$

$\Rightarrow F(x)$ converges when $|x| < 2$

& diverges when $|x| > 2$

$\Rightarrow$ Radius of conv. $R = 2$

(2) Check the endpoints $x = \pm 2$

$$F(2) = \sum_{n=0}^{\infty} \left(\frac{2}{2}\right)^n = \sum_{n=0}^{\infty} 1 \longrightarrow \text{diverges}$$

$$F(-2) = \sum_{n=0}^{\infty} \frac{(-2)^n}{2^n} = 1 - 1 + 1 - 1 + 1 - \cdots \longrightarrow \text{diverges}$$

So $F(x)$ converges when $|x| < 2$.

— x —

Example: Where does

$$F(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n} (x-5)^n \quad \text{converge?}$$

Soln: ① let $a_n = \frac{(-1)^n}{4^n n}(x-5)^n$

and let $\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$$\rho = \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{4^{n+1}(n+1)}\cdot\frac{4^n n}{(x-5)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x-5}{4}\cdot\frac{n}{n+1}\right| = \frac{|x-5|}{4}$$

To have $\rho<1$, $\frac{|x-5|}{4}<1 \iff |x-5|<4$

So $F(x)$ conv. absolutely when

$$1<x<9$$

and diverges when $x>9$ or $x<+1$

② at the endpoints:

$$F(1) = \sum_{n=1}^{\infty}\frac{(-1)^n}{4^n n}(-4)^n = \sum\frac{1}{n} \longrightarrow \text{div.}$$

$$F(9) = \sum_{n=1}^{\infty}\frac{(-1)^n}{4^n n}(4)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} \longrightarrow \text{converges}$$

(take my word for it)

Example Show that $\frac{1}{1-3x} = \sum_{n=0}^{\infty} 3^n x^n$ when $|x| < \frac{1}{3}$

Soln: Since $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$ when $|r| < 1$

we can plug in $3x$ in place of r to get

$$\frac{1}{1-3x} = \sum_{n=0}^{\infty} (3x)^n \quad \text{when } |3x| < 1$$

$$= \sum_{n=0}^{\infty} 3^n x^n \quad \text{when } |x| < \tfrac{1}{3}.$$

Example: Find a power series with center $c = 0$ for $F(x) = \frac{1}{2+x^2}$

Solution: again $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$ when $|r| < 1$

$-r = \frac{x^2}{2} \Rightarrow r = -x^2/2$

$$F(x) = \frac{1}{2} \cdot \frac{1}{1+x^2/2} = \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{x^2}{2}\right)^n$$

So $R = \sqrt{2}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{2n} \quad \text{when } \left|\frac{x^2}{2}\right| < 1 \iff |x| < \sqrt{2}$$

Differentiation & Integration

Say $F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$

has a radius of convergence R.

Then: $F(x)$ is differentiable on $(c-R, c+R)$

Furthermore, we can integrate & differentiate $F(x)$ term-by-term.

So, for $c-R < x < c+R$

$$\int F(x)\,dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-c)^{n+1}$$

and $$F'(x) = \sum_{n=0}^{\infty} n a_n (x-c)^{n-1}$$

and these series have the same radius of convergence!

Example: Find a power series for $\frac{1}{(1-x)^2}$ where $-1<x<1$. What is it's radius of conv?

Soln: $\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right)$

but $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ $\quad \downarrow d/dx$

$\Rightarrow \frac{d}{dx}\left(\frac{1}{1-x}\right) = 0 + 1 + 2x + 3x^2 + \cdots$

$\Rightarrow \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}$

& $R = 1$ (same as for $\sum x^n$

Example:

Find a power series for $\tan^{-1}x$, $-1<x<1$

Soln: We know that $\tan^{-1}x = \int_0^x \frac{1}{1+t^2}\,dt$

and $\frac{1}{1+t^2} = \frac{1}{1-(-t^2)} = \sum_{n=0}^{\infty} (-t^2)^n = \sum_{n=0}^{\infty} (-1)^n t^{2n}$

taking integrals:

$$\tan^{-1}x = \sum_{n=0}^{\infty} \underbrace{\int_0^x (-1)^n t^{2n}\,dt}_{(-1)^n \frac{x^{2n+1}}{2n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Radius of conv.: same as for $\sum_{n=0}^{\infty}(-1)^n t^{2n}$ which is $\underline{1}$

10.7 Taylor Series

Goal: We want a general method for finding Power Series

Recall: If $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ for $c-R < x < c+R$

Then we can compute the derivative of $f(x)$ by differentiating term by term.

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

$$\Rightarrow$$

$$f'(x) = 0 + a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \cdots$$

$$\Rightarrow$$

$$f''(x) = 0 + 2a_2 + (2)(3)a_3(x-c) + (3)(4)a_4(x-c)^2 + (4)(5)(x-c)^3 + \cdots$$

&

$$f'''(x) = 2\cdot 3\, a_3 + 2\cdot 3\cdot 4\, a_4(x-c) + 3\cdot 4\cdot 5\,(x-c)^2 + \cdots$$

So: $f^{(k)}(x) = k!\,a_k + 2\cdot 3\cdot 4\cdot\cdots\cdot(k-1)\,a_{k+1}(x-c) + \cdots$

k^{th} derivative

$$\Rightarrow f^{(k)}(c) = k!\,a_k \quad \text{(set } x = c \text{ above)}$$

$$\Rightarrow \boxed{a_k = f^{(k)}(c)/k!}$$

But we started with

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

So $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$$

↔ Taylor Series of $f(x)$ centered at c

that is: $$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \cdots + \frac{f^{(k)}(c)}{k!}(x-c)^k + \cdots$$

Sometimes, we may denote the Taylor series by $T(x)$.

Special Case: (Maclaurin Series)

In the special case $c=0$, $T(x)$ is called the Maclaurin Series.

Warning: If you start with a function $f(x)$ and you compute $T(x)$, there is no guarantee that $T(x)$ converges to $f(x)$. (even if $T(x)$ converges)

A^+	A	A^-	B^+	B	B^-	C^+	C	C^-	D
~~97~~	~~90~~	~~87~~	...						

e.g.: 94 86 83 ... C^+

↑ Class average

Theorem: (Taylor Series Expansion)

If $f(x)$ is represented by a power series centered at c, with $|x-c| < R$, then that power series is the Taylor series

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Example: Find the Taylor series for $f(x) = x^{-3}$ at $c=1$

Sol'n: ① Find the derivatives

$$f'(x) = -3x^{-4}$$
$$f''(x) = (-3)(-4)x^{-5}$$
$$f'''(x) = (-3)(-4)(-5)x^{-6}$$
$$\vdots$$
$$f^{(n)}(x) = (-1)^n \frac{(n+2)!}{2} x^{-(n+3)}$$

② Evaluate at c: $f^{(n)}(1) = \frac{(-1)^n (n+2)!}{2}$ and find a_n

$$a_n = \frac{f^{(n)}(1)}{n!} = \frac{(-1)^n (n+2)!}{2\, n!} = \frac{(-1)^n (n+1)(n+2)}{2}$$

③ Write $T(x)$

$$T(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)}{2} (x-1)^n$$

But how do we know if $T(x) = f(x)$?

Define the k^{th} partial sum

$$T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(c)(x-c)^n}{n!}$$

and the Remainder $R_k(x) = f(x) - T_k(x)$

$T(x)$ converges to $f(x)$ if & only if

$$\lim_{k \to \infty} R_k(x) = 0$$